SELFSIMILAR PROBLEMS OF PROPAGATION OF SHEAR CRACKS

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PMM Vol.28, № 5, 1964, pp. 889-898

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(Received April 10, 1964)

Two- and three-dimensional problems of nonsteady propagation of cracks are considered in a medium subjected to a homogeneous shear. The two-dimensional problem is completely analogous to Broberg's problem [1] of a tension crack, but is solved by a considerably simpler method. The joint investigation of the two- and three-dimensional cases also has the advantage that a number of intermediate results of the two-dimensional problem form the basis for the solution of the three-dimensional case.

The axisymmetric problem of propagation of a tension crack, the three-dimensional analogue of Broberg's problem, was solved in paper [2]. In contrast to the problem, the one which is solved in the present paper is not axisymmetrical. However, a certain generalization of the method which was applied in [2] permits construction of the exact solution of the problem at hand. It is assumed here that the surface of the crack has the form of a circular disk, i.e. that the velocity of propagation does not depend on direction. It is shown that, in general, this assumption is not borne out, but that it is possible to indicate a value of the initial stress for which the assumption is valid. For all other values of the initial stress the solution which is obtained can be considered as an approximate one.

1. Formulation of the problem. a) Two-dimensional case. A homogeneous and isotropic elastic medium having shear modulus μ and velocities of propagation of longitudinal and transverse waves a and b, respectively, fills an unbounded space and is in a state of homogeneous shear for $t < 0$, so that only one component of the stress tensor $\tau_{xz}^\circ = \tau^\circ$ is nonzero. A crack forms at the instant $t = 0$ along the y-axis, and then propagates in the plane $z = 0$ in such a way that the elastic perturbations which arise from it do not depend on the coordinate y and are polarized in the xz plane. The velocity of propagation of the crack is assumed constant and is denoted by α. The location of the crack is shown in Fig.1. The shear stresses must disappear on the surface of the crack, i.e. the perturbations caused by the development of the crack must satisfy the condition

$$\tau_{xz} = -\tau^\circ \quad \text{for} \quad z = 0, \ |x| \leqslant \alpha t$$

It can be shown that the displacement vector of the disturbance must be antisymmetric with respect to the plane $z = 0$. The shearing displacement and normal stress then prove to be odd functions z, which entails the boundary conditions for $z = 0$

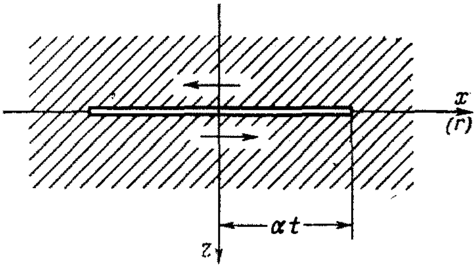

Fig. 1

$$\tau_{xz} = -\tau^{\circ} \quad \text{for } z = 0,\ |x| < \alpha t$$
$$\sigma_z = 0 \quad \text{for } z = 0,\ -\infty < x < \infty$$
$$u_x = 0 \quad \text{for } z = 0,\ |x| > \alpha t \tag{1.1}$$

Since the crack and the associated elastic disturbances are absent for $t \leqslant 0$, the initial conditions have the form ($\mathbf{u} = \{u_x, u_z\}$ is the displacement vector)

$$\mathbf{u} = 0, \qquad \mathbf{u} = \mathbf{v} = 0 \quad \text{for } t = 0$$

where the dot denotes a time derivative.

In addition to the boundary and initial conditions, it is necessary to impose a further condition on the behavior of the solution in the vicinity of the edge of the crack. As in the case of the tension crack, we shall assume that the edge of the crack is surrounded by a region on which plastic straining of the material takes place and that the dimensions of this region increase at a constant rate, proportional to the velocity of propagation of the crack. However, these dimensions are assumed to remain much smaller than those of the crack itself, so that the plastic region may be regarded as infinitesimally small. We shall also assume that the work expended in formation of the crack is proportional to the volume of the plastic region, so that the corresponding rate of doing work can be written in the form

$$w = 2\alpha^2 tC \tag{1.2}$$

where C is a constant which does not depend on α. This rate of doing work must be equal to the energy flux through a surface enclosing the edge of the crack and at an infinitesimally small distance from it. From this we obtain the required condition in the form

$$\lim_{\delta \to 0} \int_{l_\delta} \mathbf{t}_n \mathbf{v}\, dl = \alpha^2 tC \tag{1.3}$$

where the contour l_δ surrounds one of the edges of the crack and lies at a distance δ from it.

In particular, it follows from (1.3) that the stress and velocity components must increase as $\delta^{-1/2}$ with approaching the edge of the crack. It is easy to see that the components of stress and velocity are homogeneous functions of the coordinates and time of degree zero. Therefore, near the edge of the crack these must be proportional to $\sqrt{t/\delta}$.

It is clear physically that the above-mentioned plastic region must, in due course, attain some stationary size. Then in (1.3) the right-hand side becomes constant (and not proportional to time), so that the stresses in the vicinity of the edge of the crack must eventually become proportional to $\sqrt{1/\delta}$, and not to $\sqrt{t/\delta}$. That is, in due course, the self-similar character of the problem, or, in other words, the assumption of constancy of the velocity of propagation of the crack, will be violated. Thus, the above formulation of the problem is valid only for the initial stage of development of the crack.

b) Three-dimensional case. The initial state of stress of the medium is the same as in the two-dimensional case, but the propagation of the crack now brgins from the origin of coordinates. It will be assumed that the velocity of propagation of the crack is constant and does not depend on direction, so that the surface of the crack at $t > 0$ is defined by relations

$$z = 0, \qquad 0 \leqslant r < \alpha t$$

in cylindrical coordinates r, φ, z. In exactly the same way as for the two-dimensional case we reduce the problem under consideration to a boundary value problem for the half-space $z \geqslant 0$ c with the boundary conditions

$$\tau_{rz} = -\tau^\circ \cos\varphi, \qquad \tau_{\varphi z} = \tau^\circ \sin\varphi \quad \text{for} \quad z=0,\ r<\alpha t$$

$$\sigma_z = 0 \quad \text{for } z=0,\ 0 \leqslant r < \infty; \quad u_r = u_\varphi^{\cdot} = 0; \quad \text{for } z=0,\ r>\alpha t \tag{1.4}$$

and the initial conditions

$$\mathbf{u} = 0 \qquad \mathbf{u}^{\cdot} \equiv \mathbf{v} = 0, \quad \text{for } t=0 \tag{1.5}$$

We write the auxiliary condition in the form

$$\lim_{\delta\to 0} \int_{S_\delta} \mathbf{t}_n \mathbf{v}\, dS = 2\pi \alpha^3 t^2 C \tag{1.6}$$

where S_δ is a toroidal surface which surrounds the edge of the crack and is at a distance δ from it.

All the remarks made for the two-dimensional case remain in force for the three-dimensional problem as well, but now one additional fact makes its appearance. The requirement (1.6) should, strictly speaking, be formulated for the neighborhood of each point of the edge of the crack. But it would then turn out in the general case that the velocity α, which is determined by this condition, depends on the direction, i.e. the shape of the crack must differ from a circle. Unfortunately, no method is known at the present time for solving the problem when the crack boundary is an arbitraty curve. Thus, some effective value of the speed of propagation of the crack will be obtained from the condition (1.6). It should be emphasized that although this problem in its present formulation is unsuitable for determining the shape of the crack, its solution should give a correct description of the elastic wave field at long distances from the crack, which is of great importance for application of this problem in seismology. At the end of Section 4, a value of the initial stress will be indicated for which the crack is circular, even if local fulfillment of the auxiliary condition is required (for this value of the initial stress the integrand of (1.6) is independent of the angle φ).

2. Functional-invariant solutions. In both of the problems which have been formulated, the components of the stress tensor and the velocity vector are homogeneous functions of the coordinates and time of degree zero. This fact permits use of the method of the functional-invariant solutions of Smirnov and Sobolev. (*) By the use of this method it is easy to construct the solution of the two-dimensional problem for the half-space $z \geqslant 0$ with

*) See Chap. XII of the Russian translation of paper [3].

the displacement vector polarized in the xz plane and satisfying on the boundary the condition

$$\sigma_z = 0 \quad \text{for } z = 0,\ -\infty < x < \infty \tag{2.1}$$

Omitting the details, we write out this solution

$$\begin{aligned} u_x^{\cdot} &\equiv v_x = v_x^{(1)} + v_x^{(2)}, & v_x^{(1,2)} &= \operatorname{Re} V_x^{(1,2)}(\vartheta^{(1,2)}) \\ u_z^{\cdot} &\equiv v_z = v_z^{(1)} + v_k^{(2)}, & v_z^{(1,2)} &= \operatorname{Re} V_z^{(1,2)}(\vartheta^{(1,2)}) \\ \sigma_z &= \sigma_z^{(1)} + \sigma_z^{(2)}, & \sigma_z^{(1,2)} &= \operatorname{Re} \Sigma_z^{(1,2)}(\vartheta^{(1,2)}) \\ \tau_{xz} &= \tau_{xz}^{(1)} + \tau_{xz}^{(2)}, & \tau_{xz}^{(1,2)} &= \operatorname{Re} T_{xz}^{(1,2)}(\vartheta^{(1,2)}) \end{aligned} \tag{2.2}$$

Here $\vartheta^{(1)}$ and $\vartheta^{(2)}$ are determined from Equations (2.3)

$$\delta^{(1)} \equiv t - \vartheta^{(1)}x - z\sqrt{a^{-2} - \vartheta^{(1)2}} = 0, \quad \delta^{(2)} \equiv t - \vartheta^{(2)}x - z\sqrt{b^{-2} - \vartheta^{(2)2}} = 0$$

and the functions whose real parts occur in (2.2) are expressed in terms of a single unknown function $V(\vartheta)$ by the relations

$$V_x^{(1)'}(\vartheta) = 2b^2\vartheta^2 V'(\vartheta), \qquad V_x^{(2)'}(\vartheta) = (1 - 2b^2\vartheta^2)V'(\vartheta) \tag{2.4}$$

$$V_z^{(1)'}(\vartheta) = 2b^2\vartheta\sqrt{a^{-2} - \vartheta^2}\,V'(\vartheta)$$

$$V_z^{(2)'}(\vartheta) = -\vartheta(1 - 2b^2\vartheta^2)(b^{-2} - \vartheta^2)^{-1/2}V'(\vartheta)$$

$$T_{xz}^{(1)'}(\vartheta) = -4\mu b^2\vartheta^2\sqrt{a^{-2} - \vartheta^2}\,V'(\vartheta), \quad T_{xz}^{(2)'}(\vartheta) = \frac{-4\mu b^2(\vartheta^2 - {}^1/_2 b^{-2})^2}{\sqrt{b^{-2} - \vartheta^2}}V'(\vartheta)$$

$$\Sigma_z^{(1)'}(\vartheta) = -\Sigma_z^{(2)'}(\vartheta) = -2\mu\vartheta(1 - 2b^2\vartheta^2)V'(\vartheta)$$

One more solucion of a two-dimensional problem, in which the displacement vector is parallel to the y-axis, is required to construct the solution of the three-dimensional problem. This solution is determined by the relations

$$\begin{aligned} u_y^{\cdot} &\equiv v_y = v_y^{(2)}, \quad v_y^{(2)} = \operatorname{Re} V_1(\vartheta^{(2)}) \\ \tau_{yz} &= \tau_{yz}^{(2)}, \quad \tau_{yz}^{(2)} = \operatorname{Re} T_{yz}^{(2)}(\vartheta^{(2)}) \\ T_{yz}^{(2)'}(\vartheta) &= -\mu_1'\sqrt{b^{-2} - \vartheta^2}\,V_1'(\vartheta) \end{aligned} \tag{2.5}$$

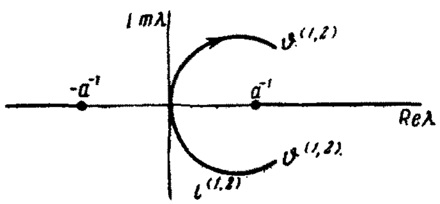

Fig. 2

In Equations (2.2) and (2.5) the imaginary instead of the real parts of the respective functions may be taken.

3. Solution of the two-dimensional problem. The relations in the preceeding Section express the derivatives of the unknown functions in terms of the derivative of the function $V(\vartheta)$. The primitives must be determined so that the initial conditions are satisfied. It is easy to see that in order to do this the integration should be carried out along the contours shown in Fig.2, so that

$$v_x^{(1)} = \frac{1}{2i} \int_{l(1)} V_x^{(1)\prime}(\lambda)\, d\lambda = -ib^2 \int_{l(1)} \lambda^2 V'(\lambda)\, d\lambda \tag{3.1}$$

$$v_x^{(2)} = \frac{1}{2i} \int_{l(2)} V_x^{(2)}(\lambda)\, d\lambda = \frac{1}{2i} \int_{l(2)} (1 - 2b^2\lambda^2) V'(\lambda)\, d\lambda \quad \text{etc.}$$

The initial conditions will be satisfied if it is required that $V'(\lambda)$ be regular for $-a^{-1} < \operatorname{Re}\lambda < a^{-1}$, and if the principal brances of the radicals $\sqrt{a^{-2} - \lambda^2}$ and $\sqrt{b^{-2} - \lambda^2}$ are chosen so that they are positive for $\lambda = 0$, which is done by making branch cuts from $-a^{-1}$ to $-\infty$ and from a^{-1} to ∞ along the real axis

The functions $\vartheta^{(1)}$ and $\vartheta^{(2)}$ assume the same values for $z = 0$, $\vartheta^{(1)} = \vartheta^{(2)} = \vartheta \equiv t / x$, and as a result we obtain

$$v_x = \frac{1}{2i}\int_l V'(\lambda)\, d\lambda, \qquad \tau_{xz} = 2i\mu b^2 \int_l \frac{R(\lambda^2)}{\sqrt{b^{-2} - \lambda^2}} V'(\lambda)\, d\lambda \qquad \text{for } z = 0 \tag{3.2}$$

where l is the contour shown in Fig.3 and

$$R(\lambda^2) = (\lambda^2 - {}^1/_2\, b^{-2})^2 + \lambda^2 (\sqrt{a^{-2} - \lambda^2}\sqrt{b^{-2} - \lambda^2}$$

The expressions (3.2) must satisfy the conditions (1.1). From these conditions and the required behavior of the solution in the vicinity of the edge of the crack we obtain, by analyzing (3.2),

$$V'(\lambda) = \frac{A}{(\alpha^{-2} - \lambda^2)^{3/2}} \qquad \text{for } 0 < \alpha < c \tag{3.3}$$

$$V'(\lambda) = \frac{A}{(\alpha^{-2} - \lambda^2)^{3/2}} + \frac{B}{(c^{-2} - \lambda^2)(\alpha^{-2} - \lambda^2)^{1/2}} \qquad \text{for } c < \alpha < b \tag{3.4}$$

where c is the Rayleigh wave velocity $(R(c^{-2}) = 0)$. In the case $b < \alpha < a$, it can be concluded that there exists no solution having the necessary variation near the edge of the crack. It can be shown that the integral on the right-hand side of (1.3) turns out to be negative in the case $c < \alpha < b$, i.e. the auxiliary condition cannot be satisfied in this case. In what follows we shall consider that $\alpha < c$. With the aid of (3.3) we obtain from the first relation of (3.2) the displacement of the sides of the crack

$$u_x = \alpha A\sqrt{\alpha^2 t^2 - x^2} \qquad \text{for } z = 0,\ |x| < \alpha t \tag{3.5}$$

The second relation of (3.2) provides an equation for the determination of A. To do this we deform the contour l, for $|\vartheta| < \alpha^{-1}$ so that it coincides with the imaginary axis, which can be done since the integrand is regular away from the cuts from $\pm a^{-1}$ to $\pm \alpha^{-1}$ and falls off sufficiently rapidly at infinity. It then follows from (1.1) and (3.2) that

$$\tau^\circ = 4\mu b^2 A \int_0^\infty \frac{(\lambda^2 + {}^1/_2 b^{-2})^2 - \lambda^2\sqrt{a^{-2} + \lambda^2}\sqrt{b^{-2} + \lambda^2}}{\sqrt{b^{-2} + \lambda^2}\,(\alpha^{-2} + \lambda^2)^{3/2}}\, d\lambda$$

We shall denote the integral in this formula by $I(\alpha)$. Then

$$A = \frac{\tau^\circ}{4\mu b^2 I(\alpha)} \tag{3.6}$$

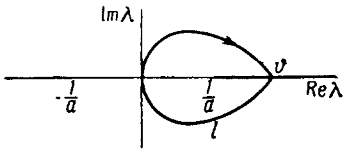

Fig. 3

It remains now to investigate the behavior of the solution near the edge of the crack and to satisfy the auxiliary condition (1.3). When the point (x, z) tends toward the edge of the crack, the ends of the contours $l^{(1)}$ and $l^{(2)}$ approach the point $\lambda = \alpha^{-1}$ (as we approach the right end of the crack). By the method of Laplace it is easy to obtain the first terms of the asymptotic expansions of the velocity and stress components in the form

$$v_x^{(1)} \approx 2b^2 A\,(\alpha t / 2\delta)^{1/2}\,\mathrm{Im}\, f^{(1)}(\psi), \qquad v_x^{(2)} \approx (\alpha^2 - 2b^2)\,(\alpha t/2\delta)^{1/2}\,\mathrm{Im}\, f^{(2)}(\psi)$$

$$v_z^{(1)} \approx 2b^2 \sqrt{1 - \alpha^2 a^{-2}}\, A\,(\alpha t / 2\delta)^{1/2}\,\mathrm{Re}\, f^{(1)}(\psi)$$

$$v_z^{(2)} \approx \frac{\alpha^2 - 2b^2}{\sqrt{1 - \alpha^2 b^{-2}}}\, A \left(\frac{\alpha t}{2\delta}\right)^{1/2} \mathrm{Re}\, f^{(2)}(\psi)$$

$$\sigma_z^{(1)} \approx 2\mu\alpha^{-1}\,(2b^2 - \alpha^2)\, A\,(\alpha t / 2\delta)^{1/2}\,\mathrm{Im}\, f^{(1)}(\psi)$$

$$\sigma_z^{(2)} \approx 2\mu\alpha^{-1}\,(\alpha^2 - 2b^2)\, A\,(\alpha t / 2\delta)^{1/2}\,\mathrm{Im}\, f^{(2)}(\psi)$$

$$\tau_{xz}^{(1)} \approx 4\mu b^2 \alpha^{-1} \sqrt{1 - \alpha^2 a^{-2}}\, A\,(\alpha t / 2\delta)\,\mathrm{Re}\, f^{(1)}(\psi)$$

$$\tau_{xz}^{(2)} \approx 4\mu b^2 \frac{(1 - \alpha^2 / 2b^2)^2}{\alpha\sqrt{1 - \alpha^2 b^{-2}}}\, A \left(\frac{\alpha t}{2\delta}\right)^{1/2} \mathrm{Re}\, f^{(2)}(\psi) \tag{3.7}$$

where

$$|x| = \alpha t + \delta\cos\psi, \quad z = \delta\sin\psi, \quad f^{(1)}(\psi) = (\cos\psi - i\sin\psi\sqrt{1 - \alpha^2 a^{-2}})^{-1/2}$$

$$f^{(2)}(\psi) = (\cos\psi - i\sin\psi\sqrt{1 - \alpha^2 b^{-2}})^{-1/2}$$

It is now easy to compute the limit of the integral on the left-hand side of (1.3). Using (3.6), we obtain as the result

$$\frac{\pi\tau^{\circ 2}}{16\mu b^2 [I(\alpha)]^2 \sqrt{1 - \alpha^2 b^{-2}}}\left[\left(1 - \frac{\alpha^2}{a^2}\right)^{1/2}\left(1 - \frac{\alpha^2}{b^2}\right)^{1/2} - \left(1 - \frac{\alpha^2}{2b^2}\right)\right] = C \tag{3.8}$$

This equation determines the velocity of propagation of the crack, α, as a function of the loading τ°.

4. Solution of the three-dimensional problem. The solution of the three-dimensional problem will be reduced to a two-dimensional one in the same way as was done in [2] for the axisymmetric problem of a tension crack. The case under consideration now differs from the previous one in that it is not axisymmetrical.

We introduce a cartesian coordinate system x, y, z depending on a parameter ω and connected to the basic polar system r, φ, z by the relations

$$x = r\cos(\varphi - \omega), \qquad y = r\sin(\varphi - \omega), \qquad z = z$$

and we form a superposition of two-dimensional solutions as follows:

$$\mathbf{u} = \int_{-\pi}^{\pi} [\mathbf{u}_1(x, z, t)\cos\omega + \mathbf{u}_2(x, z, t)\sin\omega]\, d\omega$$

where $\mathbf{u}_1(x, z, t)$ is determined from (2.2) and $\mathbf{u}_2(x, z, t)$ from (2.5). It is not difficult to see that the vector which has been obtained satisfies the equations of motion, the condition

$$\sigma_z = 0 \qquad \text{for } z = 0,\ 0 \leqslant r < \infty$$

and has the necessary dependence on φ. Performing the transformation of variable $\Omega = \varphi - \omega$, and using Equations (2.2) to (2.5), we obtain

$$u_r^{\cdot} \equiv v_r = v_r^{(1)} + v_r^{(2)}, \qquad v_r^{(1)} = \cos\varphi \int_{-\pi}^{\pi} V_x^{(1)}(\vartheta^{(1)})\cos^2\Omega\, d\Omega$$

$$v_r^{(2)} = \cos\varphi \int_{-\pi}^{\pi} [V_x^{(2)}(\vartheta^{(2)})\cos^2\Omega - V_1(\vartheta^{(2)})\sin^2\Omega]\, d\Omega$$

$$u_\varphi^{\cdot} \equiv v_\varphi = v_\varphi^{(1)} + v_\varphi^{(2)}, \qquad v_\varphi^{(1)} = -\sin\varphi \int_{-\pi}^{\pi} V_x^{(1)}(\vartheta^{(1)})\sin^2\Omega\, d\Omega$$

$$v_\varphi^{(2)} = -\sin\varphi \int_{\pi}^{\pi} [V_x^{(2)}(\vartheta^{(2)})\sin^2\Omega - V_1(\vartheta^{(2)})\cos^2\Omega]\, d\Omega$$

$$u_z^{\cdot} \equiv v_z = v_z^{(1)} + v_z^{(2)}, \qquad v_z^{(1,2)} = \cos\varphi \int_{-\pi}^{\pi} V_z^{(1,2)}(\vartheta^{(1,2)})\cos\Omega\, d\Omega$$

$$\sigma_z = \sigma_z^{(1)} + \sigma_z^{(2)}, \qquad \sigma_z^{(1,2)} = .\cos\varphi \int_{-\pi}^{\pi} \Sigma_z^{(1,2)}(\vartheta^{(1,2)})\cos\Omega\, d\Omega$$

$$\tau_{rz} = \tau_{rz}^{(1)} + \tau_{rz}^{(2)}, \qquad \tau_{rz}^{(1)} = \cos\varphi \int_{-\pi}^{\pi} T_{xz}^{(1)}(\vartheta^{(1)})\cos^2\Omega\, d\Omega \qquad (4.1)$$

$$\tau_{rz}^{(2)} = \cos\varphi \int_{-\pi}^{\pi} [T_{xz}^{(2)}(\vartheta^{(2)})\cos^2\Omega - T_{yz}^{(2)}(\vartheta^{(2)})\sin^2\Omega]\, d\Omega$$

$$\tau_{\varphi z} = \tau_{\varphi z}^{(1)} + \tau_{\varphi z}^{(2)}, \qquad \tau_{\varphi z}^{(1)} = -\sin\varphi \int_{-\pi}^{\pi} T_{xz}^{(1)}(\vartheta^{(1)})\sin^2\Omega\, d\Omega$$

$$\tau_{\varphi z}^{(2)} = -\sin\varphi \int_{-\pi}^{\pi} [T_{xz}^{(2)}(\vartheta^{(2)})\sin^2\Omega - T_{yz}^{(2)}(\vartheta^{(2)})\cos^2\Omega]\, d\Omega$$

In these expressions the functions $\vartheta^{(1,2)}$ are determined from Equations

$$\delta^{(1)} \equiv t - \vartheta^{(1)} r \cos \Omega - z\sqrt{a^{-2} - \vartheta^{(1)2}} = 0$$

$$\delta^{(2)} \equiv t - \vartheta^{(2)} r \cos \Omega - z\sqrt{b^{-2} - \vartheta^{(2)2}} = 0$$

and the integrands are expressed in terms of the two unknown functions $V(\vartheta)$ and $V_1(\vartheta)$ in accordance with (2.2) and (2.5). It is easy to see that $V(\vartheta)$ and $V_1(\vartheta)$ can be regarded as even functions of ϑ, since the terms in (4.1) which correspond to the odd parts of these functions vanish identically. We introduce the notation

$$F(\vartheta^2) = V(\vartheta), \qquad F_1(\vartheta^2) = V_1(\vartheta).$$

Taking into account that $\vartheta^{(1)} = \vartheta^{(2)} = \vartheta \equiv t / r\cos\Omega$ for $z = 0$, it is possible to obtain the following expressions from (4.1)

$$\frac{r}{2\cos\varphi} v_r^{\cdot} = \mathrm{Re}\int_{l_\nu}\left[\frac{\nu_0}{\nu}F'(\nu) - \left(1-\frac{\nu_0}{\nu}\right)F_1'(\nu)\right]\frac{d\nu}{\sqrt{\nu-\nu_0}}, \quad \nu_0 = \frac{t^2}{r^2}$$

$$-\frac{r}{2\sin\varphi} v_\varphi^{\cdot} = \mathrm{Re}\int_{l_\nu}\left[\left(1-\frac{\nu_0}{\nu}\right)F'(\nu) - \frac{\nu_0}{\nu}F_1'(\nu)\right]\frac{d\nu}{\sqrt{\nu-\nu_0}}, \quad \nu = \vartheta^2 \qquad (4.2)$$

$$-\frac{r}{2\mu\cos\varphi}\tau_{rz}^{\cdot} = \mathrm{Re}\int_{l_\nu}\left[\frac{4b^2R(\nu)}{\sqrt{b^{-2}-\nu}}F'(\nu)\frac{\nu_0}{\nu} - \left(1-\frac{\nu_0}{\nu}\right)\sqrt{b^{-2}-\nu}\,F_1'(\nu)\right]\frac{d\nu}{\sqrt{\nu-\nu_0}}$$

$$\frac{r}{2\mu\sin\varphi}\tau_{\varphi z}^{\cdot} = \mathrm{Re}\int_{l_\nu}\left[\frac{4b^2T(\nu)}{\sqrt{b^{-2}-\nu}}\left(1-\frac{\nu_0}{\nu}\right)F'(\nu) - \frac{\nu_0}{\nu}\sqrt{b^{-2}-\nu}\,F_1'(\nu)\right]\frac{d\nu}{\sqrt{\nu-\nu_0}} \qquad \text{for } z \doteq 0$$

The path of integration l_ν is shown in Fig.4. Expressions (4.2) should vanish for $\nu_0 < a^{-2}$ in order to satisfy the initial conditions. This will be met if $F'(\nu)$ and $F_1'(\nu)$ are regular, away from the branch cut, from a^{-2} to ∞, satisfy the condition

$$F'(0) = -F_1'(0) \qquad (4.3)$$

and fall off faster than ν^{-1} at infinity. By virtue of the boundary conditions, the first two expressions of (4.2) must vanish for $\nu_0 < a^{-2}$. In order for this to occur, $F'(\nu)$ and $F_1'(\nu)$ must be regular for $\mathrm{Re}\,\nu < a^{-2}$. On the other hand, the last two expressions of (4.2) must disappear for $\nu_0 > a^{-2}$. For this, the integrands must be regular for

$$\mathrm{Re}\,\nu > \nu_0 > a^{-2}.$$

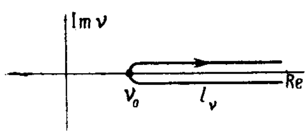

Fig. 4

We shall find $F'(\nu)$ and $F_1'(\nu)$ from these conditions and from the required behavior of the stresses in the vicinity of the edge of the crack. It can be shown that, as in the two-dimensional problem, the auxiliary condition can be satisfied only for $\alpha < c$. Then the functions $F'(\nu)$ and $F_1'(\nu)$ in the form

$$F'(\nu) = -F_1'(\nu) = \frac{A}{(\alpha^{-2}-\nu)^2}$$

satisfy the required conditions, (A is an as yet undetermined constant).

Analogously to Equation (4.2), we can obtain for $z = 0$

$$\begin{aligned}
v_r &= \sqrt{\nu_0}\cos\varphi \,\mathrm{Re}\int_{l_\nu}\left[\frac{\nu_0}{\nu}F(\nu)-\left(1-\frac{\nu_0}{\nu}\right)F_1(\nu)\right]\frac{d\nu}{\nu\sqrt{\nu-\nu_0}}\\
v_\varphi &= -\sqrt{\nu_0}\sin\varphi \,\mathrm{Re}\int_{l_\nu}\left[\left(1-\frac{\nu_0}{\nu}\right)F(\nu)-\frac{\nu_0}{\nu}F_1(\nu)\right]\frac{d\nu}{\nu\sqrt{\nu-\nu_0}}\\
\tau_{rz} &= -\sqrt{\nu_0}\cos\varphi \,\mathrm{Re}\int_{l_\nu}\left[\frac{\nu_0}{\nu}G(\nu)-\left(1-\frac{\nu_0}{\nu}\right)G_1(\nu)\right]\frac{d\nu}{\nu\sqrt{\nu-\nu_0}}\\
\tau_{\varphi z} &= \sqrt{\nu_0}\sin\varphi \,\mathrm{Re}\int_{l_\nu}\left[\left(1-\frac{\nu_0}{\nu}\right)G(\nu)-\frac{\nu_0}{\nu}G_1(\nu)\right]\frac{d\nu}{\nu\sqrt{\nu-\nu_0}}
\end{aligned}\qquad(4.4)$$

where

$$F(\nu)=\int_0^\nu F'(\lambda)\,d\lambda,\qquad F_1(\nu)=\int_0^\nu F_1'(\lambda)\,d\lambda,\qquad G(\nu)=\mu\int_0^\nu\frac{4b^2R(\lambda)}{\sqrt{b^{-2}-\lambda}}F'(\lambda)\,d\lambda$$

$$G_1(\nu)=\mu\int_0^\nu\sqrt{b^{-2}-\lambda}\,F'_1(\lambda)\,d\lambda$$

The lower limit is chosen equal to zero h re so that the integrands of (4.4) are regular for $\nu = 0$, which is necessary in order to satisfy the initial conditicns. The functions $F(\nu)$ and $F_1(\nu)$ are easily calculated; we obtain for the result

$$F(\nu) = -F_1(\nu) = \frac{\alpha^2 A\nu}{\alpha^{-2}-\nu}$$

and we find from (4.4)

$$v_r = -2\pi\cos\varphi\frac{\alpha^3 At}{\sqrt{\alpha^2t^2-r^2}},\qquad v_\varphi = 2\pi\sin\varphi\frac{\alpha^3 At}{\sqrt{\alpha^2t^2-r^2}}\qquad \text{for } z=0,\ r<\alpha t\quad(4.5)$$

We transform the expressions for $G(\nu)$ and $G_1(\nu)$ in the following way:

$$G(\nu)=4\mu b^2\int_0^{-\infty}\frac{R(\nu)}{\sqrt{b^{-2}-\nu}}F'(\nu)\,d\nu+4\mu b^2\int_{\nu\infty}^{\nu}\frac{R(\lambda)}{\sqrt{b^{-2}-\nu}}F'(\nu)\,d\lambda = M+G^*(\nu)$$

$$G_1(\nu)=\mu\int_0^{-\infty}\sqrt{b^{-2}-\nu}\,F_1'(\nu)\,d\nu+\mu\int_{\nu\infty}^{\nu}\sqrt{b^{-2}-\lambda}\,F_1'(\lambda)\,d\lambda=M_1+G_1^*(\nu)$$

The terms in (4.4) which correspond to $G^*(\nu)$ and $G_1^*(\nu)$ disappear for $\nu_0 > \alpha^{-2}$, since the integrands containing these functions are regular for $\mathrm{Re}\,\nu > \nu_0 > \alpha^{-2}$. Thus,

$$\tau_{rz} = -(M-M_1)\pi\cos\varphi,\qquad \tau_{\varphi z} = (M-M_1)\pi\sin\varphi\qquad \text{for } z=0,\ r<\alpha t$$

By comparing these values with the boundary conditions (1.4), we obtain

$$(M - M_1)\pi = \tau^\circ$$

or

$$\tau^\circ = -\mu\pi A \int_0^\infty \left[4b^2 \frac{(\nu + {}^1/_2 b^{-1})^2 - \nu\sqrt{a^{-2}+\nu}\sqrt{b^{-2}+\nu}}{\sqrt{b^{-2}+\nu}} + \sqrt{b^{-2}+\nu}\right] \frac{d\nu}{(\alpha^{-2}+\nu)^2} \qquad (4.6)$$

We denote the integral in this equation by $I_1(\alpha)$ and set $A_1 = -\pi A$. Then

$$A_1 = \frac{\tau^\circ}{\mu I_1(\alpha)} \qquad (4.7)$$

Now, integrating Equation (4.5) with respect to time, we find

$$u_r = 2A_1\alpha\cos\varphi\sqrt{\alpha^2t^2 - r^2}, \quad u_\varphi = -2A_1\alpha\sin\varphi\sqrt{\alpha^2t^2 - r^2} \quad \text{for } z = 0,\ r \leqslant \alpha \qquad (4.8)$$

or, in Cartesian coordinates

$$u_x = 2\alpha A_1\sqrt{\alpha^2t^2 - r^2}, \qquad u_y = 0 \quad \text{for } z = 0,\ r \leqslant \alpha t \qquad (4.9)$$

where $x = r\cos\varphi$, $y = r\sin\varphi$. Thus, the direction of the displacement of the edges of the crack coincides with the direction of the initial stress.

By the method which was discussed in [2], asymptotic expressions can be obtained for the velocities and stresses near the edge of the crack

$$\begin{gathered}
v_r^{(1)} \approx 2b^2A_1\cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Im}\,f^{(1)}(\psi), \\
v_r^{(2)} \approx (\alpha^2 - 2b^2)\,A_1\cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Im}\,f^{(2)}(\psi) \\
v_\varphi^{(1)} \approx O(1), \qquad v_\varphi^{(2)} \approx -\alpha^2A_1\sin\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Im}\,f^{(2)}(\psi) \\
v_z^{(1)} \approx 2b^2\sqrt{1 - \alpha^2a^{-2}}A_1\cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Re}\,f^{(1)}(\psi) \\
v_z^{(2)} \approx (\alpha^2 - 2b^2)(1 - \alpha^2b^{-2})^{-1/2}A_1\cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Re}\,f^{(2)}(\psi) \\
\sigma_z^{(1)} \approx -2\mu\alpha^{-1}(\alpha^2 - 2b^2)\,A_1\cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Im}\,f^{(1)}(\psi) \\
\sigma_z^{(2)} \approx 2\mu\alpha^{-1}(\alpha^2 - 2b^2)\,A_1\cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Im}\,f^{(2)}(\psi) \\
\tau_{rz}^{(1)} \approx -4\mu b^2\alpha^{-1}\sqrt{1 - \alpha^2a^{-2}}A_1\cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Re}\,f^{(1)}(\psi) \\
\tau_{rz}^{(2)} \approx \mu b^{-2}\alpha^{-1}(1 - \alpha^2b^{-2})^{-1/2}(\alpha^2 - 2b^2)^2A_1\cos\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Re}\,f^{(2)}(\psi) \\
\tau_{\varphi z}^{(1)} \approx O(1), \qquad \tau_{\varphi z}^{(2)} \approx \mu\alpha\sqrt{1 - \alpha^2b^{-2}}A_1\sin\varphi\,(2\alpha t/\delta)^{1/2}\,\mathrm{Re}\,f^{(2)}(\psi)
\end{gathered} \qquad (4.10)$$

where $r = \alpha t + \delta\cos\psi$, $z = \delta\sin\psi$, and the functions $f^{(1)}(\psi)$ and $f^{(2)}(\psi)$ are the same as in (3.7). With the aid of these expressions we obtain from the condition (1.6), the following relation:

$$\mu A_1 \int_0^{2\pi} \left\{ \frac{4b^2\left[\sqrt{1 - \alpha^2a^{-2}}\sqrt{1 - \alpha^2b^{-2}} - (1 - \alpha^2/2b^2)^2\right]}{\sqrt{1 - \alpha^2b^{-2}}}\cos^2\varphi + \right.$$
$$\left. + \alpha^2\sqrt{1 - \alpha^2b^{-2}}\sin^2\varphi \right\} d\varphi = C \qquad (4.11)$$

It is evident from this that, as was noted during the formulation of the problem, the velocity α must, strictly speaking, be a function of φ, since in the general case the integrand depends on φ for constant α. However, there exists one value of α, namely,

$$\alpha = \alpha_1 = b\left(1 - \frac{a^2}{9a^2 - 16b^2}\right)^{1/2} < c \qquad (4.12)$$

for which the integrand in (4.11) is seen to be constant, since in this case the coefficients of $\cos^2\varphi$ and $\sin^2\varphi$ are the same. For this value of α, the crack will have a circular shape, The corresponding value of the initial stress is obtained from (4.7), (4.11), and (4.12)

$$\tau^\circ(\alpha_1) = \alpha_1^{-1} I_1(\alpha_1)\left[\frac{\mu}{2\pi} C\left(1 - \frac{\alpha_1^2}{b^2}\right)\right]^{1/2} \qquad (4.13)$$

Performing the integration with respect to φ in (4.11) in the general case, we obtain Equation

$$\frac{\pi\tau^{\circ 2}}{\mu\,[I_1(\alpha)]^2}\left\{\frac{4b^2}{\sqrt{1-\alpha^2 b^{-2}}}\left[\sqrt{1-\alpha^2 a^{-2}}\sqrt{1-\alpha^2 b^{-2}} - \left(1 - \frac{\alpha^2}{2b^2}\right)^2\right] + \alpha^2\sqrt{1-\alpha^2 b^{-2}}\right\} = C \qquad (4.14)$$

which determines some effective value of α. It is clear that the closer τ° is to the value in (4.13), the better will the solution which has been constructed describe the phenomenon of propagation of the crack.

The author is grateful to N.V. Zvolinskii for his interest in this work.

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Translated by A.R.R.